Knot Groups and Colorability

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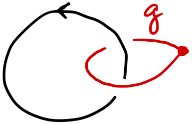
Knot Group

"Knots are determined by their complement"

Gordon-Luecke Theorem: If two knots have complements that are orientation-preserving homeomorphic then they are isotopic.

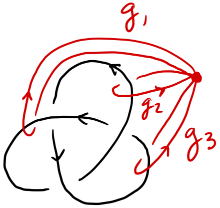
Wirtinger Presentation

Example: Unknot



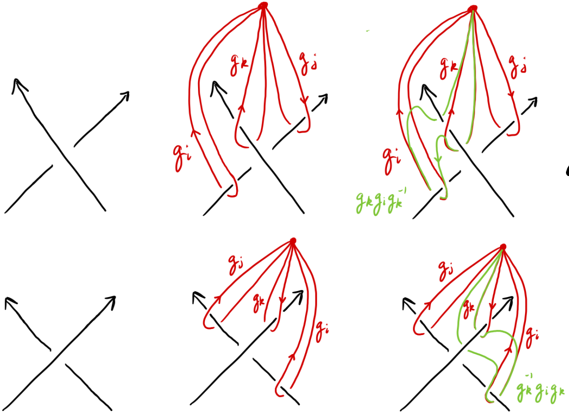
$\langle g | \rangle \cong \mathbb{Z}$

Example: Trefoil



Wirtinger Presentation

Relators in the Wirtinger presentation

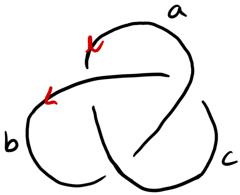


$$g_k g_i g_k^{-1} g_j^{-1} = 1$$

$$g_k^{-1} g_i g_k g_j^{-1} = 1$$

Wirtinger Presentation

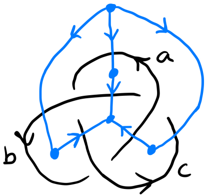
Example: Trefoil



$$\langle a, b, c | a^{-1}cab^{-1}, b^{-1}abc^{-1}, c^{-1}bca^{-1} \rangle$$

Wirtinger Presentation

Example: Trefoil



$$\langle a, b, c | a^{-1}cab^{-1}, b^{-1}abc^{-1}, c^{-1}bca^{-1} \rangle$$

$$cab^{-1}a^{-1} \cdot abc^{-1}b^{-1} = cab^{-1}bc^{-1}b^{-1} = cac^{-1}b^{-1}$$
$$ac^{-1}b^{-1}c = (c^{-1}bca^{-1})^{-1}$$

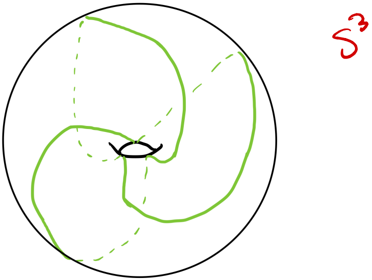
so $c^{-1}bca^{-1} \in \langle a^{-1}cab^{-1}, b^{-1}abc^{-1} \rangle^N$.

Other Route to a Presentation

Example: Trefoil

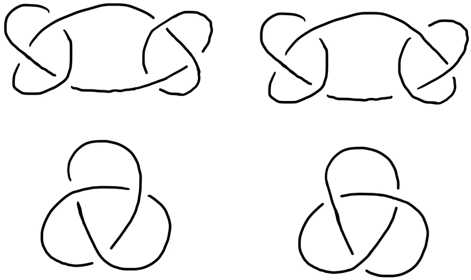
Wirtinger presentation: $\langle a, b, c | a^{-1}cab^{-1}, b^{-1}abc^{-1} \rangle$

Using Seifert-Van Kampen: $\langle x, y | x^2 = y^3 \rangle$



Knot Group as a Knot Invariant

There exist distinct knots with the same knot group.

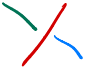


Theorem (Whitten): Prime knots in S^3 with isomorphic groups have homeomorphic complements.

Tricolorability

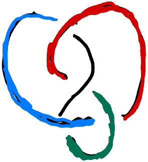
Let D be a knot diagram of the knot K. A knot is **tricolorable** if each of the strands in D can be assigned a color from a set of three colors such that

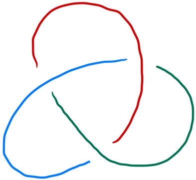
- every crossing of D uses either all one color or all distinct colors
- more than one color is used



Nonexample: Unknot

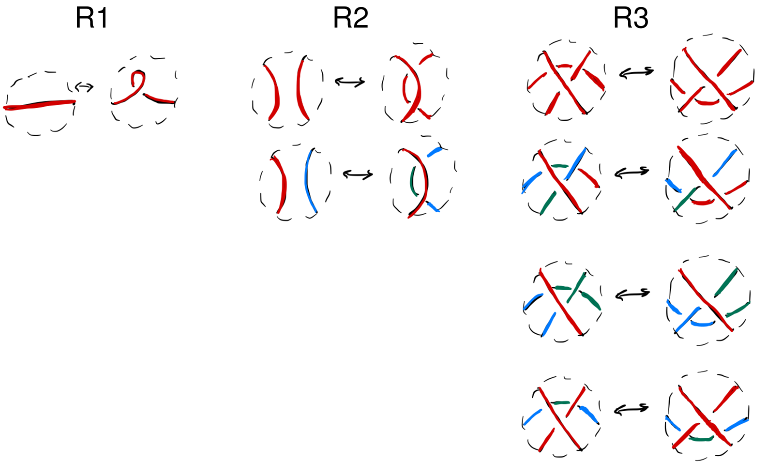
Example: Trefoil





Tricolorability is a Well-Defined Invariant

Reidemeister Moves:



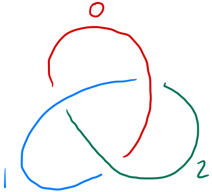
3-colorability

Equivalent definition to tricolorability: Let D be a knot diagram of the knot K. Label the strands of D $a_1, \ldots, a_m$. A knot is **3-colorable** if there exists a map $f : \{a_1, \ldots, a_m\} \to \mathbb{Z}/3\mathbb{Z}$ such that

- for all crossings in D we have $f(a_i) + f(a_j) \equiv 2f(a_k) \mod 3$ where a_i and a_j are the understrands and a_k is the overstrand
- $|f\{a_1, \ldots, a_m\}| > 1$

$a + a \equiv 2a$

Example: Trefoil



$0 + 2 \equiv 2 \cdot 1 \mod 3$

$2 + 1 \equiv 0 \mod 3$

$1 + 0 \equiv 2 \cdot 2 \mod 3$

n-colorability

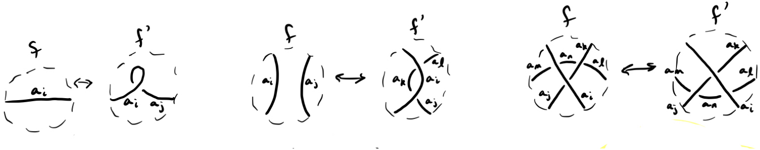
Let D be a knot diagram of the knot K. Label the strands of D $a_1, ..., a_m$. A knot is ***n*-colorable** if there exists a map $f : \{a_1, ..., a_m\} \rightarrow \mathbb{Z}/n\mathbb{Z}$ such that

- for all crossings in D we have $f(a_i) + f(a_j) \equiv 2f(a_k) \mod n$ where a_i and a_j are the understrands and a_k is the overstrand
- $|f\{a_1, ..., a_m\}| > 1$

n-colorability is a Well-Defined Invariant

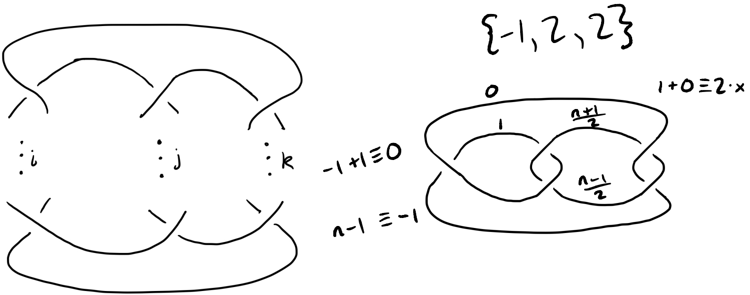
Reidemeister Moves:

R1 R2 R3



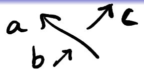
Examples of *n*-colorable Knots

Coloring pretzel ~~knots~~ links



Theorem (Butler, Cohen, Dalton, Louder, Rettberg, Whitt)
The $\{-i, 2i, 2i\}$ pretzel knot is *n*-colorable for all *n*.

Knot Group and n-colorability- Notation



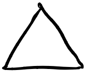
Let D be an oriented diagram of a knot K. Let $Q = \{a_1, ..., a_m\}$ be the strands of D and let $T \subset Q^3$ be the set of triples (a, b, c) corresponding to the crossings of D such that a is the overstrand, b is the incoming understrand, and c is the outgoing understrand. Then the Wirtinger presentation of the knot group of K from D is

$$\pi_1(S^3 \setminus K) = \langle Q | \{a^{-1}bac^{-1} | (a, b, c) \in T\} \rangle.$$

The dihedral group of order $2n$ has presentation

$$\mathbb{D}_n = \langle x, y | x^n = 1, y^2 = 1, yxy = x^{-1} \rangle.$$

$\mathbb{D}_3$



Knot Group and n-colorability

Theorem (Fox, Butler, Cohen, Dalton, Louder, Rettberg, Whitt) The diagram D is n-colorable if and only if there exists a homomorphism $\phi : \pi_1(S^3 \setminus K) \to \mathbb{D}_n$ such that $\phi(a_i) = yx^{k(a_i)}$ where $k : Q \to \mathbb{Z}/n\mathbb{Z}$.

$$\varphi : \langle Q \mid \{a^{-1}bac^{-1} \mid (a,b,c) \in T\} \rangle \longrightarrow \mathbb{D}_n$$

Proof

For reference: $\mathbb{D}_n = \langle x, y | x^n = 1, y^2 = 1, yxy = x^{-1} \rangle$.
($\Rightarrow$) Assume that there exists an n-coloring $k : Q \to \mathbb{Z}/n\mathbb{Z}$ of D.
$(a,b,c) \in T$

$$\begin{aligned} 2k(a) &\equiv k(b) + k(c) \mod n \\ x^{k(b)-2k(a)} &= x^{-k(c)} \\ x^{k(b)-k(a)} y x^{k(a)} y &= y x^{k(c)} y \\ x^{k(b)-k(a)} y x^{k(a)} &= y x^{k(c)} \\ x^{-k(a)} y^2 x^{k(b)} y x^{k(a)} &= y x^{k(c)} \\ (y x^{k(a)})^{-1} y x^{k(b)} y x^{k(a)} &= y x^{k(c)} \\ \phi(a^{-1} b a) &= \phi(c) \end{aligned}$$

So ϕ is a group homomorphism.

Group Theory and Knot Theory

"Given a class of groups G, with which knots, if any, does there exist a homomorphism from the knot group to a group in G?"

Reference

Explorations into Knot Theory: Colorability
by Butler, Cohen, Dalton, Louder, Rettberg, and Whitt.